Quaternions

John C. Hart

CS 418

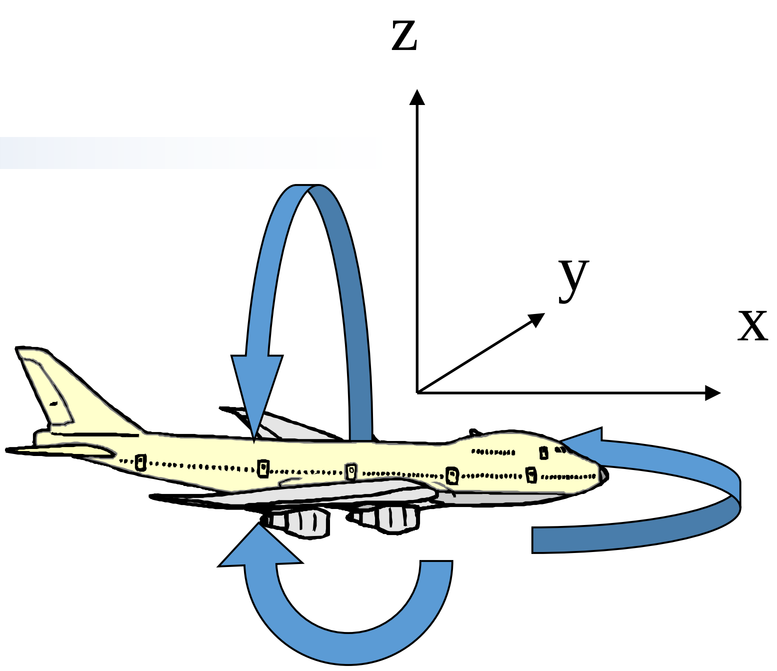
Interactive Computer Graphics

Rigid Body Dynamics

- Rigid bodies
  - Inflexible
  - Center of gravity
  - Location in space
  - Orientation in space
- Rigid body dynamics
  - Force applied to object relative to center of gravity
  - Rotation in space about center of gravity
- Orientation of a rigid body is a rotation from a fixed canonical coordinate frame
- Representing orientation = representing rotation

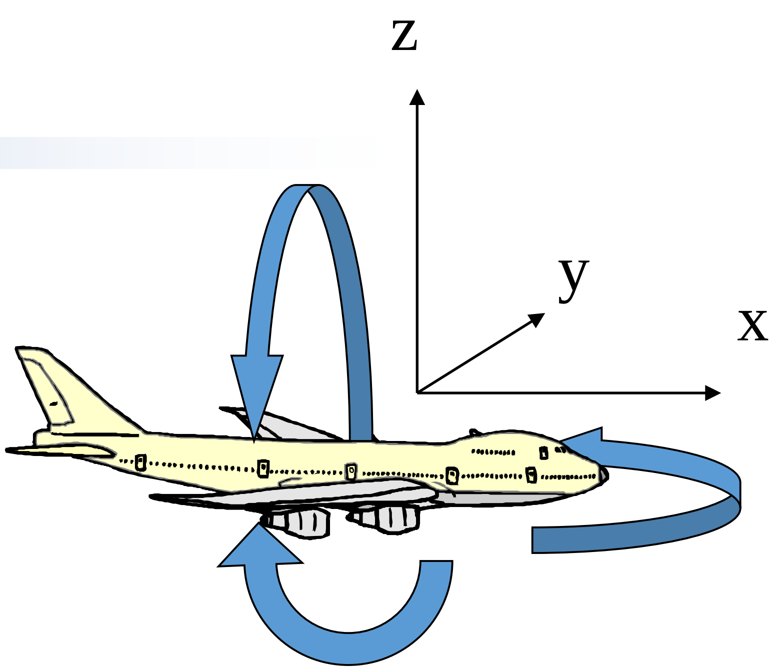
Euler Angles

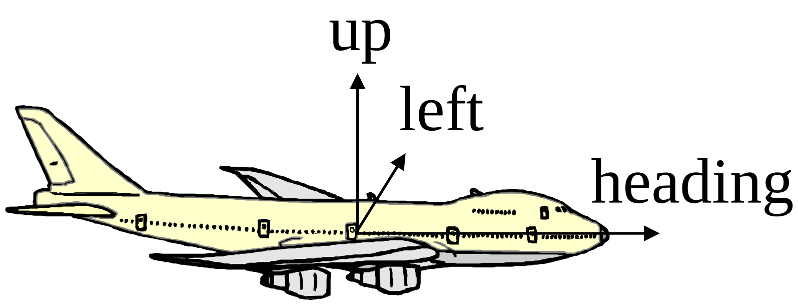
- Airplane orientation
  - Roll
    - rotation about x
    - Turn wheel
  - Pitch
    - rotation about y
    - Push/pull wheel
  - Yaw
    - rotation about z
    - Rudder (foot pedals)
- Airplane orientation
  - Rx(roll) Ry(pitch) Rz(yaw)



Local v. Global

- Roll 90 followed by pitch 90
- Which direction is plane heading?
  - In the y direction?
  - Or in the z direction?
- Depends on whether axes are local or global
- Airplane axes are local
  - heading, left, up
- Need an orientation to represent airplane coordinate system
- Orientation needs to be global





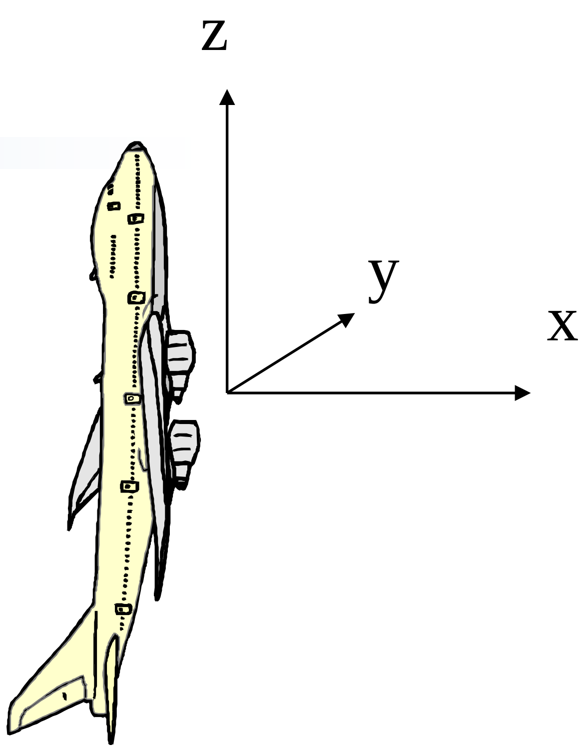
Gimbal Lock

- Airplane orientation

  Rx(roll) Ry(pitch) Rz(yaw)

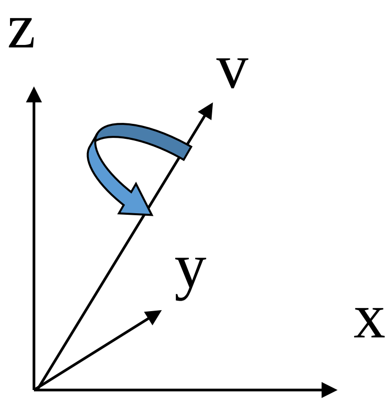
- When plane pointing up (pitch = 90), yaw is meaningless, roll direction becomes undefined

- Two axes have collapsed onto each other



Space of Orientations

- Any rotation Rx Ry Rz can be specified by a single rotation by some angle about some line through the origin
- Proof: Rx, Ry and Rz are special unitary
  - Columns (and rows) orthogonal
  - Columns (and rows) unit length
  - Product also special unitary
  - Thus product is a rotation
- Represent orientation by rotation axis (unit vector, line through origin) and rotation angle (scalar)

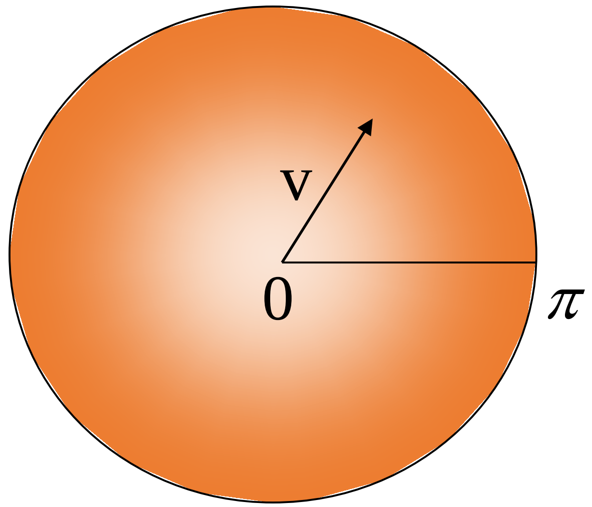


Orientation Ball

- Vector **v** represents orientation

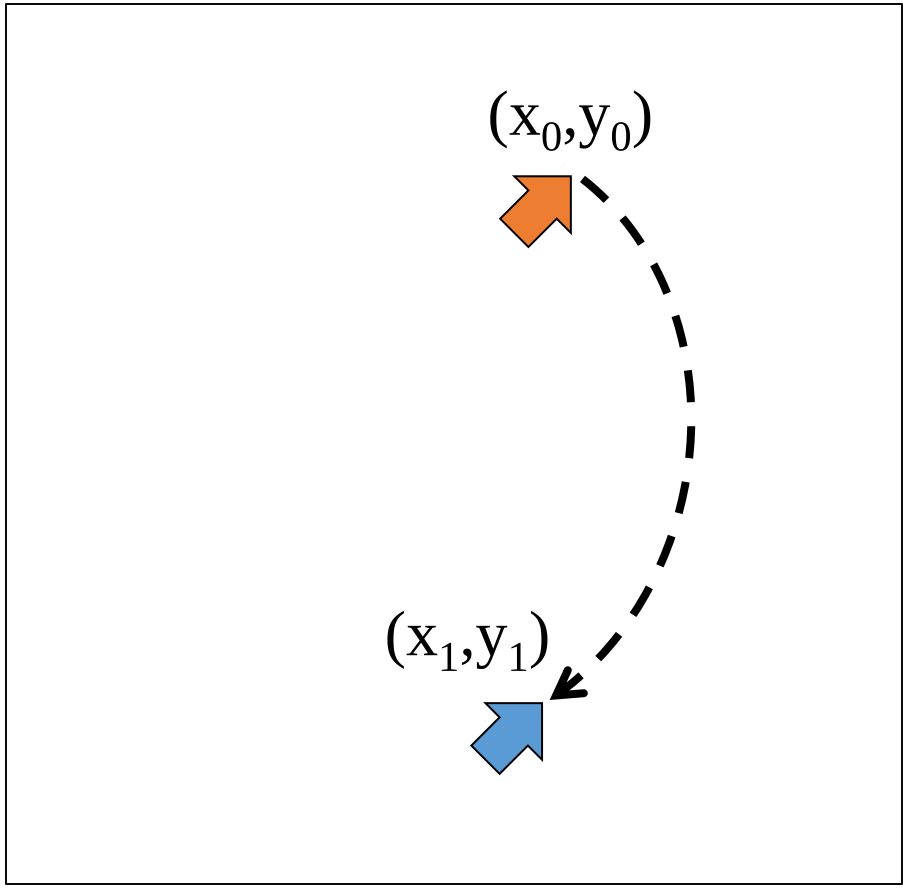
$$\|\mathbf{v}\| \leq \pi$$

- Decompose $\mathbf{v} = \theta\,\mathbf{u}$
  - θ = angle of rotation: $0 \leq \theta \leq \pi$
  - $\mathbf{u}$ = axis of rotation: $\|\mathbf{u}\| = 1$
- Angles greater than π represented by $-\mathbf{u}$
- All orientations represented by a point in the orientation ball



ArcBall

- How to rotate something on the screen?
- Assume canvas (window) coordinates
- Click one point (x_0,y_0)
- Drag to point (x_1,y_1)



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- Click one point (x_0,y_0)
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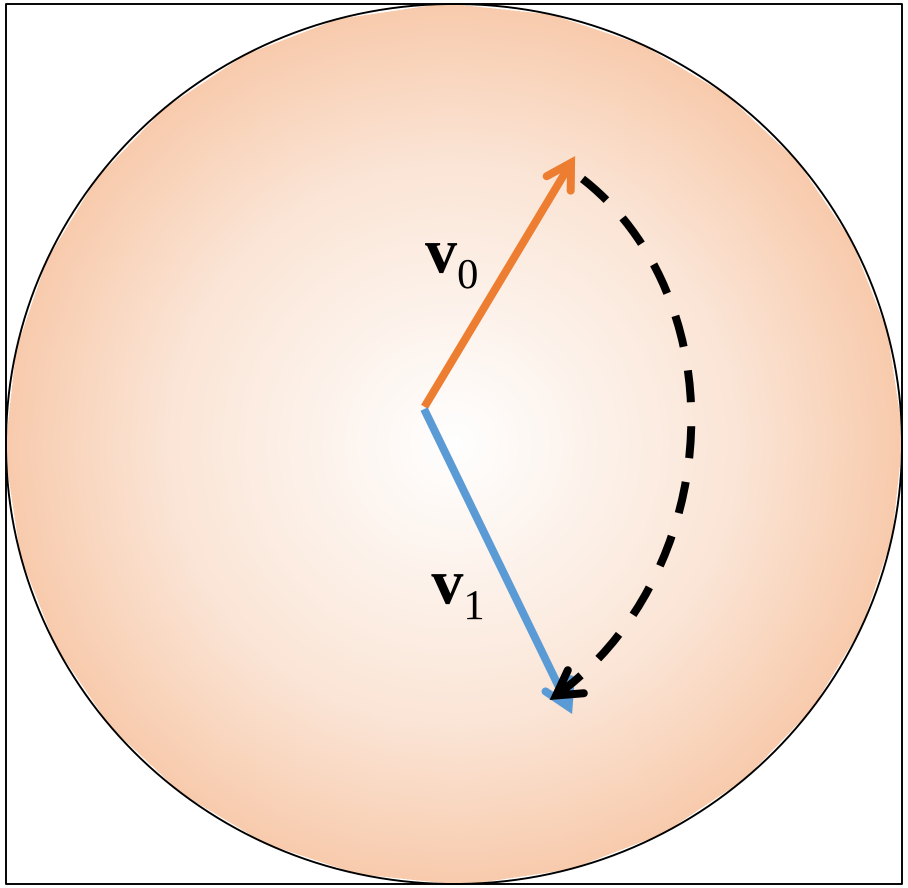
- Consider sphere over screen
- Then $z_0 = \mathrm{sqrt}(1 - x_0^2 - y_0^2)$ and $z_1 = \mathrm{sqrt}(1 - x_1 - y_1)$ give points on sphere $\mathbf{v}_0$ and $\mathbf{v}_1$.



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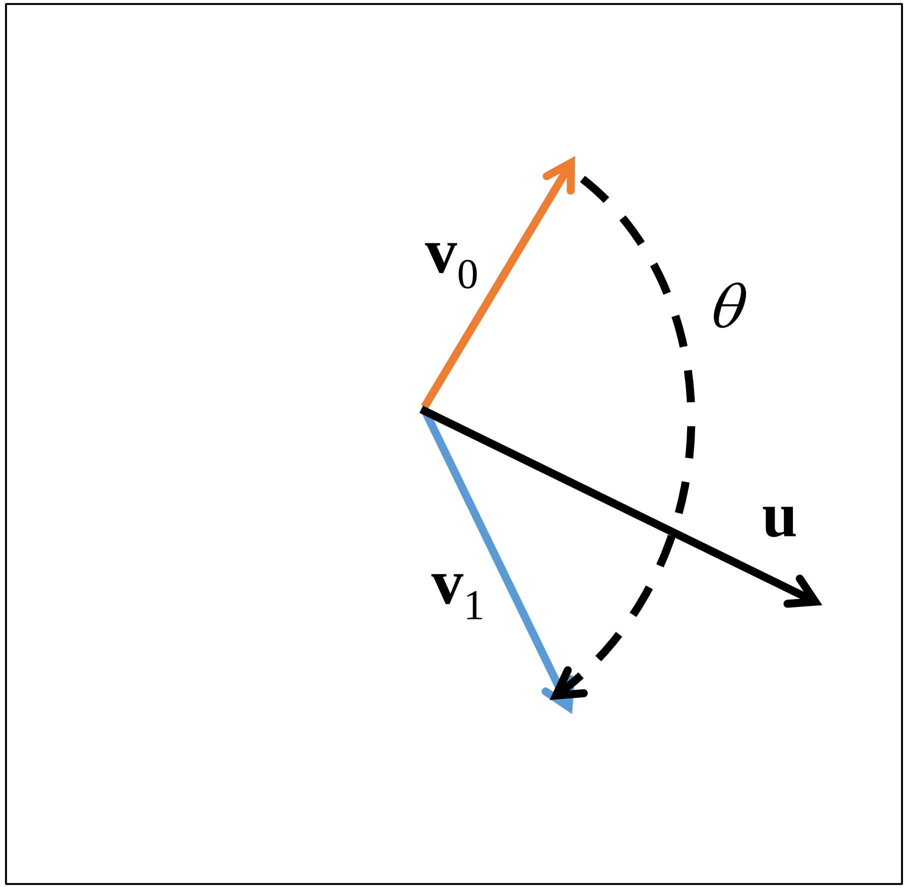
- Then rotation axis is $\mathbf{u} = (\mathbf{v}_0 \times \mathbf{v}_1)/||\mathbf{v}_0 \times \mathbf{v}_1||$
- Angle is $\theta = \sin^{-1} ||\mathbf{v}_0 \times \mathbf{v}_1||$



Quaternions

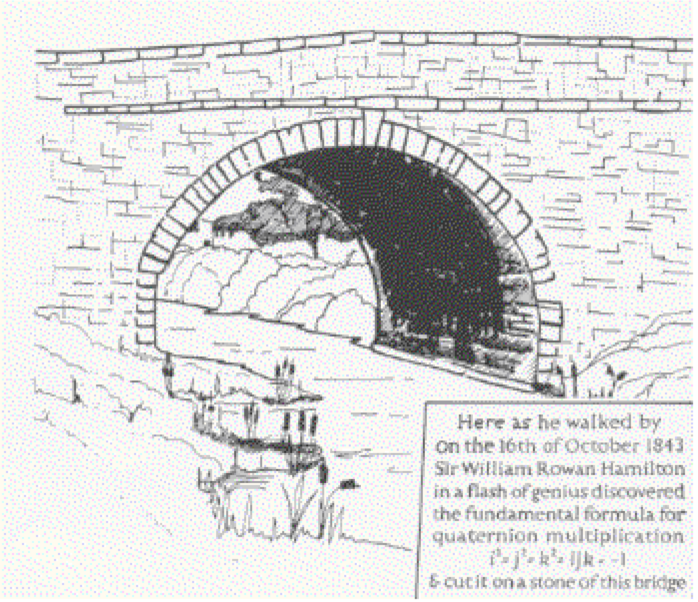
- Quaternions are 4-D numbers

$$q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$

- With one real axis
- And three imaginary axes: **i,j,k**
- Imaginary multiplication rules

$$\mathbf{ij} = \mathbf{k},\ \mathbf{jk} = \mathbf{i},\ \mathbf{ki} = \mathbf{j}$$

$$\mathbf{ji} = -\mathbf{k},\ \mathbf{kj} = -\mathbf{i},\ \mathbf{ik} = -\mathbf{j}$$



Hamilton Math Inst.,
Trinity College

Quaternion Multiplication

$$(a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) \times (a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k})$$

$$\begin{aligned} = \quad & a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 + \\ & (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\,\mathbf{i} + \\ & (a_1c_2 + c_1a_2 + d_1b_2 - b_1d_2)\,\mathbf{j} + \\ & (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)\,\mathbf{k} \end{aligned}$$

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- Scalar, vector pair: q = $(a,\mathbf{v})$, where $\mathbf{v} = (b,c,d) = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$
- Multiplication combines dot and cross products

$$q_1\, q_2 = (a_1,\mathbf{v}_1)\,(a_2,\mathbf{v}_2) = (a_1a_2 - \mathbf{v}_1\cdot\mathbf{v}_2,\ a_1\mathbf{v}_2 + a_2\mathbf{v}_1 + \mathbf{v}_1\times\mathbf{v}_2)$$

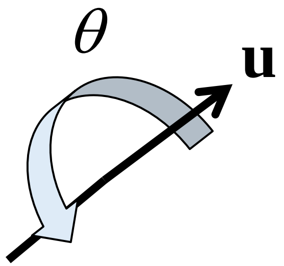
Unit Quaternions

- Length: $|q|^2 = a^2 + b^2 + c^2 + d^2 = 1$
  - Unit quaternions are points on the unit 3-sphere
  - Space of orientations is a three dimensional manifold, but one embedded in four dimensional space

Unit Quaternions

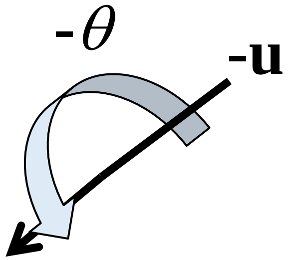
- Length: $|q|^2 = a^2 + b^2 + c^2 + d^2$
- Let $q = \cos(\theta/2) + \sin(\theta/2)\,\mathbf{u}$ be a unit quaternion: $|q| = |\mathbf{u}| = 1$.
  - $\mathbf{u} = b\mathbf{i} + c\mathbf{j} + \mathbf{d}k$
  - the 3-sphere is a "double cover" of the space of orientations because the same rotation corresponds to two different unit quaternions

$$q = \cos\frac{\theta}{2} + \mathbf{u}\sin\frac{\theta}{2}$$



$\cos(\theta/2) + \sin(\theta/2)\,\mathbf{u}$

$\cos(-\theta/2) + \sin(-\theta/2)\,(-\mathbf{u})$



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- Let point $\mathbf{p} = (x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

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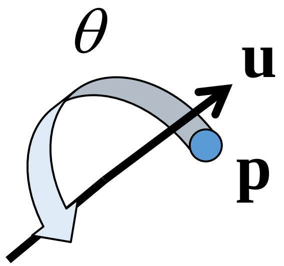

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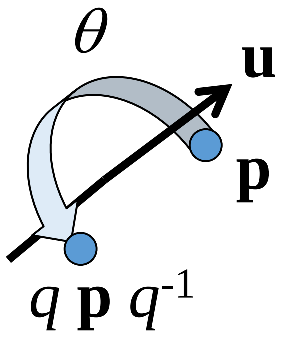

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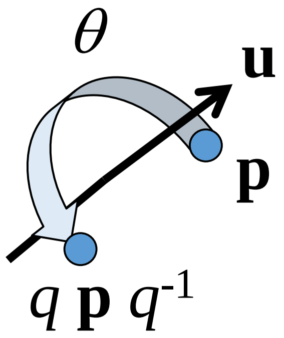
- Inverse of a unit quaternion is its conjugate (negate the imaginary part)

$$q^{-1} = (\cos(\theta/2) + \sin(\theta/2)\,\mathbf{u})^{-1}$$
$$= \cos(-\theta/2) + \sin(-\theta/2)\,\mathbf{u}$$
$$= \cos(\theta/2) - \sin(\theta/2)\,\mathbf{u}$$

$$q = \cos\frac{\theta}{2} + \mathbf{u}\sin\frac{\theta}{2}$$



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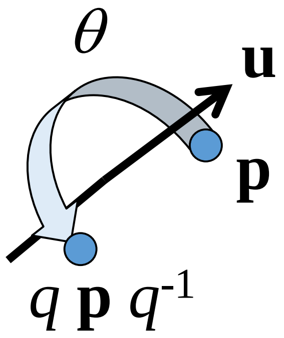
- Composition of rotations

$$q_{21} = q_2\, q_1 \neq q_1\, q_2$$

$$q = \cos\frac{\theta}{2} + \mathbf{u}\sin\frac{\theta}{2}$$



$$q_{21}\, \mathbf{p}\, q_{21}^{-1}$$
$$(q_2\, q_1)\, \mathbf{p}\, (q_1^{-1}\, q_2^{-1})$$
$$q_2\, (q_1\, \mathbf{p}\, q_1^{-1})\, q_2^{-1}$$

Quaternion to Matrix

The unit quaternion

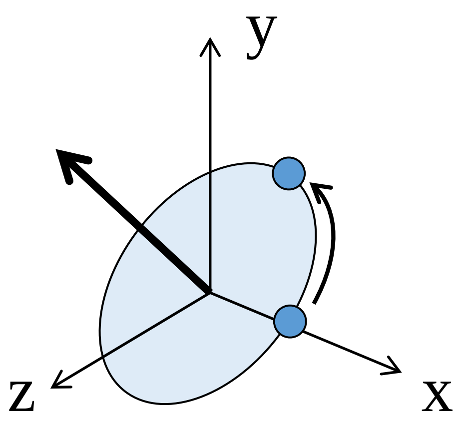
$q = a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$

corresponds to the rotation matrix

$$\begin{bmatrix} a^2+b^2-c^2-d^2 & 2bc-2ad & 2bd+2ac & \\ 2bc+2ad & a^2-b^2+c^2-d^2 & 2cd-2ab & \\ 2bd-2ac & 2cd+2ab & a^2-b^2-c^2+d^2 & \\ & & & 1 \end{bmatrix}$$

Example

Rotate the point **p** = (1,0,0) by 90° about the axis (0,.707,.707)



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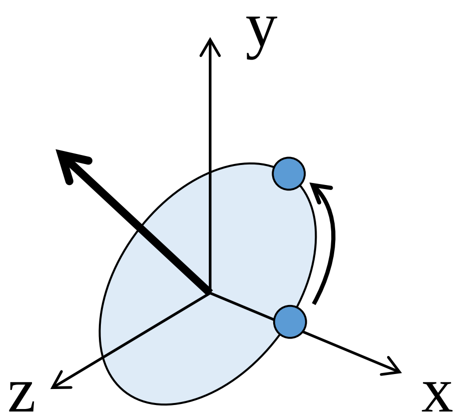

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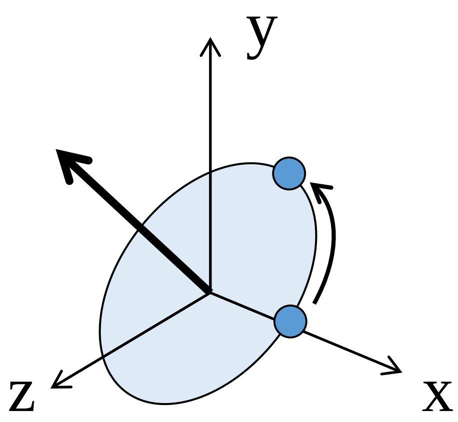

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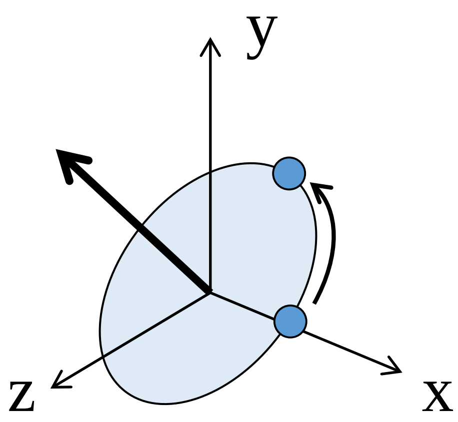

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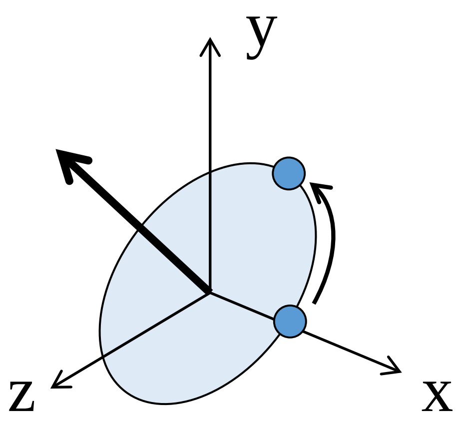

$$q\ p\ q^{-1} = (.707 + .5j + .5k)(i)(.707 - .5j - .5k)$$

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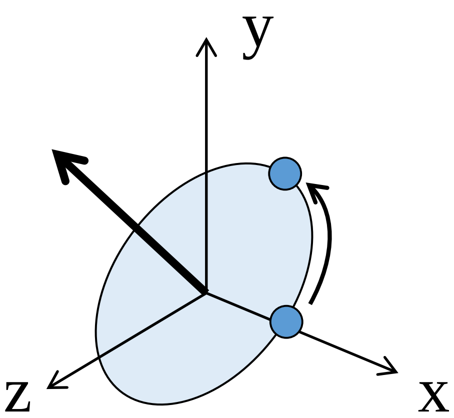

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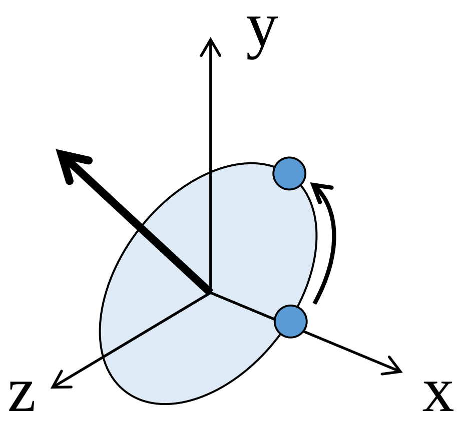

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$$= (.5i - .354k + .354j) + (-.354k - .25i - .25) + (.354j + .25 - .25i)$$

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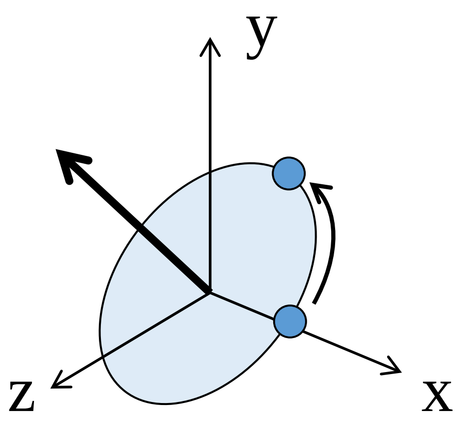

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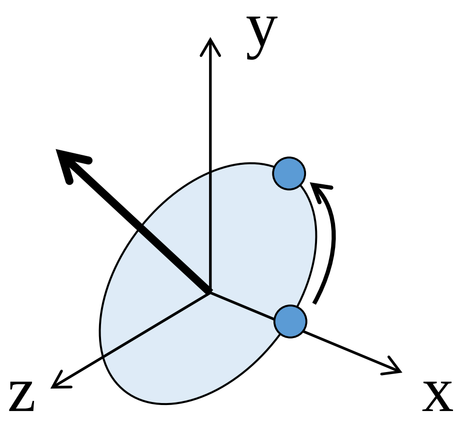

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$$= 0 + (.5 - .25 - .25)i + (.354 + .354)j + (-.354 - .354)k$$
$$= .707j - .707k$$

Exponential Map

- Recall complex numbers

$$e^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta$$

- Quaternion $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ can be written like a complex number $a + \beta\mathbf{u}$ where $\beta = \|(b,c,d)\|$ and $\mathbf{u}$ is a unit pure quaternion $\mathbf{u} = (b\mathbf{i} + c\mathbf{j} + d\mathbf{k})/\|(b,c,d)\|$

- Exponential map for quaternions

$$e^{\mathbf{u}\theta} = \cos\theta + \mathbf{u}\sin\theta$$

- Quaternion that rotates by θ about $\mathbf{u}$ is

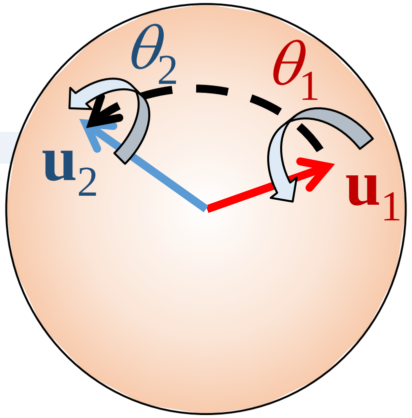
$$q = e^{\theta/2\,\mathbf{u}} = \cos\theta/2 + \sin\theta/2\,\mathbf{u}$$

SLERP

- Interpolating orientations requires a “straight line” between unit quaternion orientations on the 3-sphere
- The canonical orientation is a zero degree rotation represented by the unit quaternion $1 + 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$
- We can interpolate from the canonical orientation to a given orientation $(\theta,\mathbf{u})$ as

$$q(t) = \cos t\theta/2 + \mathbf{u} \sin t\theta/2 = e^{t\,(\theta/2)\mathbf{u}}$$

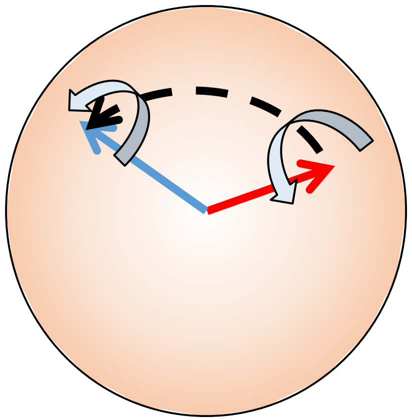
SLERP



- To interpolate from q_1 to q_2
  we can interpolate from the base

$$q(t) = q_1(q_1^{-1}q_2)^t$$
$$= \exp((\theta_1/2)\,\mathbf{u}_1)\,(\exp((-\theta_1/2)\,\mathbf{u}_1)\,\exp((\theta_2/2)\,\mathbf{u}_2)\,)^t$$
$$= \exp((\theta_1/2)\,\mathbf{u}_1 + t((-\theta_1/2)\,\mathbf{u}_1) + t((\theta_2/2)\,\mathbf{u}_2))$$
$$= \exp((1-t)(\theta_1/2)\,\mathbf{u}_1 + t((\theta_2/2)\,\mathbf{u}_2))$$

- Rotation interpolation is the exponential map of a linear interpolation between points in the orientation ball



= e